

Calculation of the energy levels and sizes of baryons with a noncentral harmonic potential

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PACS numbers: 12.40.Yx Hadron mass models and calculations. 12.39.Jh Non-relativistic quark model. 12.39.Pn Potential Models

Abstract. It is considered that the effective interaction between any two quarks in a baryon can be approximately described by a simple harmonic potential. Also, it is made use of the nonrelativistic approximation since the effective (constituent) masses of quarks are not small. The problem is firstly solved in Cartesian coordinates in order to find the energy levels irrespective of their angular momenta and then it is also solved in polar cylindrical coordinates for taking into account the angular momenta of the levels. By making a comparison between the two solutions the energies and the corresponding angular momenta (and parity) of almost all baryon levels are described. The agreement with the experimental data is quite impressive. The solution in Cartesian coordinates also produces some very important figures for the sizes of baryons and for the harmonic oscillator constant which is clearly related to confinement.

1 Introduction

As is well known there are several important works that deal with the calculation of the energy levels of baryons. One of the most important ones is the pioneering work of Gasiorowicz and Rosner [1] which has calculation of baryon levels and magnetic moments of baryons using approximate wavefunctions. Another important work is that of Isgur and Karl [2] which strongly suggests that non-relativistic quantum mechanics can be used in the calculation of baryon spectra. Other very important attempts towards the understanding of baryon spectra are the works of Capstick and Isgur [3], Bhaduri et al.[4], Murthy et al.[5], Murthy et al.[6], and Stassat et al.[7]. An important work attempting to describe baryon spectra is the recent work of Hosaka, Toki and Takayama [8] published in 1998. This last work arrives at an important equation which had already been deduced by De Souza a long time ago, in 1992 [9]. Other works by De Souza published before 1998 include it [10,11].

The effective potential between any two quarks is not known and because of this several different potentials are found in the literature. In particular the harmonic approximation using a harmonic central potential has been widely used. Since the three quarks of a baryon are always in a plane it is assumed that the effective potential between any two quarks of a baryon can be given by a linear harmonic oscillator. The motion of the plane of quarks is not considered in this paper. This is a calculation quite different from those found in the literature.

2 Calculation in Cartesian coordinates

In the initial calculation we use normal cartesian coordinates which, of course, does not consider the angular momentum of the system, that is, it does not take into account the symmetries of the system. But this section is very important because it calculates the energy levels. In the next section we will link each level to its angular momentum. Considering the work of Isgur and Karl [2] as to the use of non-relativistic quantum mechanics and using a linear harmonic oscillator potential [8,9] we can write the Hamiltonian in normal cartesian coordinates as

$$\sum_{i=1}^6 \partial^2 \psi \partial \xi_i^2 + 2\hbar^2 \left(E - 12 \sum_{i=1}^6 \omega_i \xi_i^2 \right) \psi = 0 \quad (1)$$

where we have used the fact that the three quarks are always in a plane. The above equation may be resolved into a sum of 6 equations

$$\partial^2 \psi \partial \xi_i^2 + 2\hbar^2 (E_i - 12\omega_i \xi_i^2) \psi = 0, \quad (2)$$

which is the equation of a single harmonic oscillator of potential energy $12\omega_i \xi_i^2$ and unitary mass with $E = \sum_{i=1}^6 E_i$.

The general solution is a superposition of 6 harmonic motions in the 6 normal coordinates. The eigenfunctions $\psi_i(\xi_i)$ are the ordinary harmonic oscillator eigenfunctions

$$\psi_i(\xi_i) = N_{v_i} e^{-(\alpha_i/2)\xi_i^2} H_{v_i}(\sqrt{\alpha_i}\xi_i), \quad (3)$$

where N_{v_i} is a normalization constant, $\alpha_i = \nu_i/\hbar$ and $H_{v_i}(\sqrt{\alpha_i}\xi_i)$ is a Hermite polynomial of the v_i th degree. For large ξ_i the eigenfunctions are governed by the exponential functions which make the eigenfunctions go to zero very fast.

The energy of each harmonic oscillator is

$$E_i = h\nu_i(v_i + 12), \quad (4)$$

where $v_i = 0, 1, 2, 3, \dots$ and ν_i is the classical oscillation frequency of the normal “vibration” i , and v_i is the “vibrational” quantum number. The total energy of the system can assume only the values

$$E(v_1, v_2, v_3, \dots, v_6) = h\nu_1(v_1 + 12) + h\nu_2(v_2 + 12) + \dots h\nu_6(v_6 + 12). \quad (5)$$

As was said above the three quarks in a baryon must always be in a plane. Therefore, each quark is composed of two oscillators and so we may rearrange

the energy expression as

$$E(n, m, k) = h\nu_1(n + 1) + h\nu_2(m + 1) + h\nu_3(k + 1), \quad (6)$$

where $n = v_1 + v_2, m = v_3 + v_4, k = v_5 + v_6$. Of course, n, m, k can assume the values, 0,1,2,3,... We may find the constants $h\nu$ from the ground states of some baryons. They are the known quark constituent masses taken as $m_u = m_d = 0.31\text{Gev}$, $m_s = 0.5\text{Gev}$, $m_c = 1.7\text{Gev}$, $m_b = 5\text{Gev}$ and $m_t = 174\text{GeV}$.

The states obtained with the above Hamiltonian are degenerate with respect to isospin so that our calculation does not distinguish between nucleonic and Δ states, or between Σ and Λ states. In the tables below the experimental values of baryon masses were taken from reference 12.

Let us start the calculation with the states ddu (neutron), uud (proton) and $ddd(\Delta^-)$, $uuu(\Delta^{++})$ and their resonances. All the energies below are given in Gev. Because $m_u = m_d$, we have that the energies calculated by the formula

$$E_{n,m,k} = 0.31(n + m + k + 3) \quad (7)$$

correspond to many energy states. The calculated values are displayed in Table 1. The last column on the right is a rough classification which will be cleared up in the next section.

The energies of the particles Λ and Σ , which are composed of uus and uds are given by

$$E_{n,m,k} = 0.31(n + m + 2) + 0.5(k + 1). \quad (8)$$

The results are displayed in Table 2. The agreement with the experimental values is excellent. For the $\Xi^o(uss)$ and $\Xi^-(dss)$ baryons the energies are expressed by

$$E_{n,m,k} = 0.31(n + 1) + 0.5(m + k + 2). \quad (9)$$

See Table 3 to check the agreement with the experimental data. In this case the last column is almost empty due to a lack of experimental data.

In the same way the energies of $\Omega(sss)$ are obtained by

$$E_{n,m,k} = 0.5(n + m + k + 3). \quad (10)$$

The energies are displayed in Table 4. The discrepancies are higher, of the order of 10% and decreases as the energy increases. This is a tendency which is also observed for the other particles. This may mean that, at the bottom, the potential is less flat than the potential of a harmonic oscillator.

The energies of the charmed baryons($C = +1$) Λ_c^+ , Σ_c^{++} , Σ_c^+ and Σ_c^0 are given by

$$E_{n,m,k} = 0.31(n + m + 2) + 1.7(k + 1). \quad (11)$$

The levels are shown in Table 5.

For the charmed baryons($C = +1$) Ξ_c^+ and Ξ_c^0 we have

$$E_{n,m,k} = 0.31(n + 1) + 0.5(m + 1) + 1.7(k + 1). \quad (12)$$

The results are displayed in Table 6.

As for the Ω_c^0 , its energies are

$$E_{n,m,k} = 0.5(n + m + 2) + 1.7(k + 1). \quad (13)$$

Table 7 shows the results of the energy levels.

In all tables below E_C is the calculated value, E_M is the measured or experimental value and the Error is

$$Error = |E_M - E_C| \times 100\%.$$

As is easily seen in each table many levels may be predicted.

n, m, k	$E_C(\text{Gev})$	$E_M(\text{Gev})$	Error(%)	$L_{2I,2J}$	Parity
0,0,0	0.93	0.938(N)	0.9	P_{11}	+
$n + m + k = 1$	1.24	1.232(Δ)	0.6	P_{33}	+
$n + m + k = 2$	1.55	1.52(N)	1.9	D_{13}	-
$n + m + k = 2$	1.55	1.535(N)	1.0	S_{11}	-
$n + m + k = 2$	1.55	1.6(Δ)	3.1	P_{33}	+
$n + m + k = 2$	1.55	1.62(Δ)	4.5	S_{31}	-
$n + m + k = 3$	1.86	1.90(N)	2.2	P_{13}	+
$n + m + k = 3$	1.86	1.90(Δ)	2.2	S_{31}	-
$n + m + k = 3$	1.86	1.905(Δ)	2.4	F_{35}	+
$n + m + k = 3$	1.86	1.91(Δ)	2.7	P_{31}	+
$n + m + k = 3$	1.86	1.92(Δ)	3.2	P_{33}	+
$n + m + k = 4$	2.17	2.08(N)	4.1	D_{13}	-
$n + m + k = 4$	2.17	2.09(N)	3.7	S_{11}	-
$n + m + k = 4$	2.17	2.10(N)	3.2	P_{11}	+
$n + m + k = 4$	2.17	2.15(Δ)	0.9	S_{31}	-
$n + m + k = 4$	2.17	2.19(N)	0.9	G_{17}	-
$n + m + k = 4$	2.17	2.20(N)	1.4	D_{15}	-
$n + m + k = 4$	2.17	2.20(Δ)	1.4	G_{37}	-
$n + m + k = 4$	2.17	2.22(N)	2.3	H_{19}	+
$n + m + k = 4$	2.17	2.225(N)	5.5	G_{19}	-
$n + m + k = 5$	2.48	2.39(Δ)	3.6	F_{37}	+
$n + m + k = 5$	2.48	2.40(Δ)	3.2	G_{39}	-
$n + m + k = 5$	2.48	2.42(Δ)	2.4	$H_{3,11}$	+
$n + m + k = 6$	2.79	2.7(N)	3.2	$K_{1,13}$	+
$n + m + k = 6$	2.79	2.75(Δ)	1.4	$I_{3,13}$	-
$n + m + k = 7$	3.10	3.100(N)	0	$L_{1,15}$?
$n + m + k = 8$	3.21	?	?	?	?
$n + m + k = 9$	3.72	?	?	?	?
$n + m + k = 9$	4.03	?	?	?	?
...

Table 1. Baryon states N and Δ . The energies E_C were calculated according to the formula $E_{n,m,k} = 0.31(n + m + k + 3)$ in which n, m, k are integers. E_M is the measured energy. The error means the absolute value of $(E_C - E_M)/E_C$. We are able, of course, to predict the energies of many other resonances.

State(n, m, k)	$E_C(\text{Gev})$	$E_M(\text{Gev})$	Error(%)	$L_{2I,2J}$	Parity
0,0,0	1.12	1.116(Λ)	0.4	P_{01}	+
0,0,0	1.12	1.193(Σ)	6.5	P_{11}	+
$n + m = 1, k=0$	1.43	1.385(Σ)	3.2	P_{13}	+
$n + m = 1, k=0$	1.43	1.405(Λ)	1.7	S_{01}	-
$n + m = 1, k=0$	1.43	1.48(Σ)	3.5	?	?
0,0,1	1.62	1.52(Λ)	6.2	D_{03}	-
0,0,1	1.62	1.56(Σ)	3.7	?	?
0,0,1	1.62	1.58(Σ)	2.5	D_{13}	-
0,0,1	1.62	1.60(Λ)	1.2	P_{01}	+
0,0,1	1.62	1.62(Σ)	0	S_{11}	-
0,0,1	1.62	1.66(Σ)	2.5	P_{11}	+
0,0,1	1.62	1.67(Σ)	3.1	D_{13}	-
0,0,1	1.62	1.67(Λ)	3.1	S_{01}	-
$n + m = 2, k=0$	1.74	1.69(Λ)	2.9	D_{03}	-
$n + m = 2, k=0$	1.74	1.69(Σ)	2.9	?	?
$n + m = 2, k=0$	1.74	1.75(Σ)	0.6	S_{11}	-
$n + m = 2, k=0$	1.74	1.77(Σ)	1.7	P_{11}	+
$n + m = 2, k=0$	1.74	1.775(Σ)	2.0	D_{15}	-
$n + m = 2, k=0$	1.74	1.80(Λ)	3.4	S_{01}	-
$n + m = 2, k=0$	1.74	1.81(Λ)	4.0	P_{01}	+
$n + m = 2, k=0$	1.74	1.82(Λ)	4.6	F_{05}	+
$n + m = 2, k=0$	1.74	1.83(Λ)	5.2	D_{05}	-

Continues on next page

State(n, m, k)	$E_C(\text{Gev})$	$E_M(\text{Gev})$	Error(%)	$L_{2I,2J}$	Parity
$n + m = 1, k=1$	1.93	1.84(Σ)	4.7	P_{13}	+
$n + m = 1, k=1$	1.93	1.88(Σ)	2.6	P_{11}	+
$n + m = 1, k=1$	1.93	1.89(Λ)	2.1	P_{03}	+
$n + m = 1, k=1$	1.93	1.915(Σ)	0.8	F_{15}	+
$n + m = 1, k=1$	1.93	1.94(Σ)	0.5	D_{13}	-
$n + m = 3, k=0$	2.05	2.00(Λ)	2.5	?	-
$n + m = 3, k=0$	2.05	2.00(Σ)	2.4	S_{11}	-
$n + m = 3, k=0$	2.05	2.02(Λ)	1.5	F_{07}	+
$n + m = 3, k=0$	2.05	2.03(Σ)	1.0	F_{17}	+
$n + m = 3, k=0$	2.05	2.07(Σ)	1.0	F_{15}	+
$n + m = 3, k=0$	2.05	2.08(Σ)	1.5	P_{13}	+
0,0,2	2.12	2.10(Σ)	0.9	G_{17}	-
0,0,2	2.12	2.10(Λ)	0.9	G_{07}	-
0,0,2	2.12	2.11(Λ)	0.5	F_{05}	+
$n + m = 2, k=1$	2.24	2.25(Σ)	0.5	?	?
$n + m = 4, k=0$	2.36	2.325(Λ)	1.5	D_{03}	-
$n + m = 4, k=0$	2.36	2.35(Λ)	0.4	H_{09}	+
$n + m = 1, k=2$	2.43	2.455	2.5	?	
$n + m = 3, k=1$	2.55	2.585(Λ)	1.4	?	?
0,0,3	2.62	2.62(Σ)	0	?	?
$n + m = 5, k=0$	2.67	to be found	?	?	
$n + m = 2, k=2$	2.74	to be found	?	?	
$n + m = 4, k=1$	2.86	to be found	?	?	
$n + m = 1, k=3$	2.93	to be found	?	?	
$n + m = 6, k=0$	2.98	3.00(Σ)	0.7	?	?
$n + m = 3, k=2$	3.05	to be found	?	?	
$n = m = 0, k=4$	3.12	to be found	?	?	
$n + m = 5, k=1$	3.17	3.17(Σ)	0	?	?
$n + m = 2, k=3$	3.24	to be found	?	?	

Table 2. Baryon states Σ and Λ . The energies E_C were calculated according to the formula $E_{n,m,k} = 0.31(n + m + 2) + 0.5(k + 1)$. E_M is the measured energy. The error means the absolute value of $(E_C - E_M)/E_C$.

State(n, m, k)	E_C (Gev)	E_M (Gev)	Error(%)	$L_{2I,2J}$	Parity
0,0,0	1.31	1.315	0.5	P_{11}	+
1,0,0	1.62	1.53	5.6	P_{13}	+
1,0,0	1.62	1.62	0	?	?
1,0,0	1.62	1.69	4.3	?	?
n=0, $m+k=1$	1.81	1.82	0.6	D_{13}	-
2,0,0	1.93	1.95	1.0	?	?
n=1, $m+k=1$	2.12	2.03	4.2	?	?
n=1, $m+k=1$	2.12	2.12	0	?	?
n=3, $m=k=0$	2.24	2.25	0.5	?	?
n=0, $m+k=2$	2.31	2.37	2.6	?	?
n=2, $m+k=1$	2.43	to be found	?	?	?
n=4, $m=k=0$	2.55	2.5	2.0	?	?
n=1, $m+k=2$	2.62	to be found	?	?	?
...

Table 3. Baryon states Ξ . The energies E_C were calculated according to the formula $E_{n,m,k} = 0.31(n+1) + 0.5(m+k+2)$. E_M is the measured energy. The error means the absolute value of $(E_C - E_M)/E_C$. The state $\Xi(1530)P_{13}$ appears to be the lowest state of the composite $\Xi \oplus \pi$. Its decay is in fact $\Xi\pi$.

State(n, m, k)	E_C (Gev)	E_M (Gev)	Error(%)
0,0,0	1.5	1.672	11.7
$n+m+k=1$	2.0	2.25	12.5
$n+m+k=2$	2.5	2.47	1.2
$n+m+k=3$	3.0	to be found	?
...

Table 4. Baryon states Ω . The energies E_C were calculated according to the formula $E_{n,m,k} = 0.5(n+m+k+3)$, and E_M is the measured energy.

State(n, m, k)	E_C (Gev)	E_M (Gev)	Error(%)
0,0,0	2.32	2.285(Λ_c)	1.5
$n + m = 1, k=0$	2.63	2.594(Λ_c)	0.1
$n + m = 1, k=0$	2.63	2.627(Λ_c)	0.01
$n + m = 2, k=0$	2.94	to be found	?
...

Table 5. Baryon states Λ_c and Σ_c . The energies E_C were calculated according to the formula $E_{n,m,k} = 0.31(n + m + 2) + 1.7(k + 1)$. The state with energy 2.63 MeV had already been predicted in another version of this work. The experimental levels 2.594 MeV and 2.627 MeV have confirmed the theoretical values. It appears that the level $\Sigma_c(2.455)$ is a composition of the level (0, 0, 0)(that is the 2.285 Λ_c) with a pion as is also inferred from its decay.

State(n, m, k)	E_C (Gev)	E_M (Gev)	Error(%)
0,0,0	2.51	2.47(Ξ_c^+)	1.6
	2.51	2.47(Ξ_c^0)	1.6
1,0,0	2.82	to be found	?
0,1,0	3.01	to be found	?
...

Table 6. Baryon states Ξ_c . The energies E_C were calculated according to the formula $E_{n,m,k} = 0.31(n + 1) + 0.5(m + 1) + 1.7(k + 1)$. E_M is the measured energy. The recently found level $\Xi_c(2645)$ is probably a composition of the regular level Ξ_c^+ with a pion as its decay confirms.

State(n, m, k)	E_C (Gev)	E_M (Gev)	Error(%)
0,0,0	2.7	$2.704(\Omega_c^0)$	0
$n + m = 1, k=0$	3.2	to be found	?
$n + m = 2, k=0$	3.7	to be found	?
...

Table 7. Baryon states Ω_c . The energies E_C were calculated according to the formula $E_{n,m,k} = 0.5(n + m + 2) + 1.7(k + 1)$. The energy of the level (0, 0, 0) above shown had been predicted in other versions of this work.

We may predict the energy levels of many other baryons given by the formulas:

- ucc and dcc, $E_{n,m,k} = 0.31(n + 1) + 1.7(m + k + 2)$;
- scc, $E_{n,m,k} = 0.5(n + 1) + 1.7(m + k + 2)$;
- ccc, $E_{n,m,k} = 1.7(n + m + k + 3)$;
- ccb, $E_{n,m,k} = 1.7(n + m + 2) + 5(k + 1)$;
- cbb, $E_{n,m,k} = 1.7(n + 1) + 5(m + k + 2)$;
- ubb and dbb, $E_{n,m,k} = 0.31(n + 1) + 5(m + k + 2)$;
- uub, udb and ddb, $E_{n,m,k} = 0.31(n + m + 2) + 5(k + 1)$;
- bbb, $E_{n,m,k} = 5(n + m + k + 3)$;
- usb and dsb, $E_{n,m,k} = 0.31(n + 1) + 0.5(m + 1) + 5(k + 1)$;
- sbb, $E_{n,m,k} = 0.5(n + 1) + 5(m + k + 2)$;
- scb, $E_{n,m,k} = 0.5(n + 1) + 1.7(m + 1) + 5(k + 1)$;
- ucb, $E_{n,m,k} = 0.31(n + 1) + 1.7(m + 1) + 5(k + 1)$;
- tt, $E_{n,m,k} = (174 \pm 17)(n + m + k + 3)$;
- and all combinations of t with u, d, c, s and b.

3 Calculation in polar cilindrical coordinates

In order to address the angular momentum and parity we have to use spherical or polar coordinates. Since the three quarks of a baryon are always in a plane we can use polar coordinates. We choose the Z axis perpendicular to this plane. Now the eigenfunctions are angular momentum eigenfunctions (of the orbital angular momentum). Thus, we have three oscillators in a plane. Considering that they are independent the radial Schrödinger equation for the stationary states of each oscillator is given by [13]

$$[-\hbar^2 2\mu (\partial^2 \partial \rho^2 + 1 \rho \partial \partial \rho - m_z \rho^2) + 12\mu \omega^2 \rho^2] R_{Em}(\rho) = E R_{Em}(\rho) \quad (14)$$

where m_z is the quantum number associated to L_z . Therefore, what we have is the following: three independent oscillators with orbital angular momenta L_1 , L_2 and L_3 which have the Z components L_{z1} , L_{z2} and L_{z3} in the plane containing the quarks. Of course, the system has a total orbital angular momentum $L = L_1 + L_2 + L_3$ and there is a quantum number l_i associated to each L_i . The eigenvalues of the energy are given by¹³

$$E = (2r_1 + |m_1| + 1)h\nu_1 + (2r_2 + |m_2| + 1)h\nu_2 + (2r_3 + |m_3| + 1)h\nu_3 \quad (15)$$

in which $r_1, r_2, r_3 = 0, 1, 2, 3, \dots$ and $|m_i| = 0, 1, 2, 3, \dots, l_i$. Comparing the above equation with the equation

$$E(n, m, k) = h\nu_1(n + 1) + h\nu_2(m + 1) + h\nu_3(k + 1),$$

we see that $n = 2r_1 + |m_1|$, $m = 2r_2 + |m_2|$, $k = 2r_3 + |m_3|$.

Let us recall that if we have three angular momenta L_1, L_2 and L_3 described by the quantum numbers l_1, l_2, l_3 the total orbital angular momentum L will be described by the quantum number l given by

$$l_1 + l_2 + l_3 \geq l \geq ||l_1 - l_2| - l_3| \quad (16)$$

where $l_1 \geq |m_1|$, $l_2 \geq |m_2|$, $l_3 \geq |m_3|$.

Taking into account spin we form the total angular momentum given by $J = L + S$ and the quantum numbers of J are $j = l \pm s$ where s is the spin quantum number. As we will see we will be able to describe almost all baryon levels.

3.1 Baryons N and Δ

Let us begin the calculation with the particles N and Δ . We will classify the levels by energy according to Table 1. The first state of N is the state ($n = 0, m = 0, k = 0$) with energy 0.93 GeV. Therefore in this case $l_1 = l_2 = l_3 = 0$ and then $l = 0$. Hence this is the positive parity state P_{11} and we have

l	N	Δ	Parity
0	$0.938P_{11}$?	+

The second energy level (1.24 GeV) which is the first state of Δ has $n + m + k = 1$. This means that $2r_1 + |m_1| + 2r_2 + |m_2| + 2r_3 + |m_3| = 1$. Thus, $|m_1| + |m_2| + |m_3| = 1$ and $l_1 + l_2 + l_3 \geq 1$, and we can choose the sets $|m_1| = 1, |m_2| = |m_3| = 0$; $|m_1| = |m_3| = 0, |m_2| = 1$; $|m_1| = 1, |m_2| = |m_3| = 0$, and $l_1 = 2, l_2 = l_3 = 0$, or $l_2 = 2, l_1 = l_3 = 0$, or still $l_3 = 2, l_1 = l_2 = 0$ which produce $l = 2$ and thus the level

l	N	Δ	Parity
2	?	$1.232P_{33}$	+

In the third energy level (1.55 GeV) $n + m + k = 2 = 2r_1 + |m_1| + 2r_2 + |m_2| + 2r_3 + |m_3|$. This means that $|m_1| + |m_2| + |m_3| = 2, 0$ and we have the sets of possible values of l_1, l_2, l_3

l_1, l_2, l_3	2,0,0	0,2,0	0,0,2	1,1,0	1,0,1	0,1,1	0,0,0
l	2	2	2	0,1,2	0,1,2	0,1,2	0

in which the second column presents the values of l that satisfy the condition $l_1 + l_2 + l_3 \geq 2, 0$. There are thus the following states

l	N	Δ	Parity
0	$1.44P_{11}, 1.71P_{11}$?	+
1	$1.535S_{11}, 1.65S_{11}$ $1.52D_{13}, 1.70D_{13}, 1.675D_{15}$	$1.62S_{31}$ $1.70D_{33}$	-
2	$1.68F_{15}, 1.72P_{13}$	$1.6P_{33}$	+

because we can have $j = 1/2 = 0 + 1/2 = 1 - 1/2$; $j = 3/2 = 1 + 1/2 = 2 - 1/2$; $j = 5/2 = 1 + 3/2 = 2 + 1/2$.

The fourth energy level (1.86 GeV) has $n + m + k = 3 = 2r_1 + |m_1| + 2r_2 + |m_2| + 2r_3 + |m_3|$ which makes $|m_1| + |m_2| + |m_3| = 3, 1$ and $l_1 + l_2 + l_3 \geq 3, 1$. We have therefore the possibilities

l_1, l_2, l_3	l	l_1, l_2, l_3	l	l_1, l_2, l_3	l
3,0,0	3	2,0,1	3,2,1	0,2,1	3,2,1
0,3,0	3	1,0,2	3,2,1	1,0,0	1
0,0,3	3	1,2,0	3,2,1	0,1,0	1
2,1,0	3,2,1	0,1,2	3,2,1	0,0,1	1

and the states

l	N	Δ	Parity
1	$2.08D_{13}$	$1.90S_{31}$ $1.94D_{33}$	-
2	$1.90P_{13}, 2.00F_{15}, 1.99F_{17}$ $2.00F_{35}, 1.95F_{37}$	$1.91P_{31}, 1.92P_{33}, 1.905F_{35},$?	+
3	?	$1.93D_{35}$	-

In the fifth energy level (2.17 GeV) $n+m+k=4=2r_1+|m_1|+2r_2+|m_2|+2r_3+|m_3|$ which yields $|m_1|+|m_2|+|m_3|=4, 2, 0$ and $l_1+l_2+l_3 \geq 4, 2, 0$. We can then have $l_1=l_2=l_3=0$ ($l=0$) and also

l_1, l_2, l_3	l	l_1, l_2, l_3	l
4,0,0	4	2,2,0	4,3,2,1,0
0,4,0	4	2,0,2	4,3,2,1,0
0,0,4	4	0,2,2	4,3,2,1,0
3,1,0	4,3,2	2,0,0	2
3,0,1	4,3,2	0,2,0	2
1,3,0	4,3,2	0,0,2	2
1,0,3	4,3,2	1,1,0	2,1,0
0,3,1	4,3,2	1,0,1	2,1,0
0,1,3	4,3,2	0,1,1	2,1,0

and hence the states

l	N	Δ	Parity
0	$2.10P_{11}$?	+
1	$2.08S_{11}, 2.20D_{15}$	$2.15S_{31}, 2.35D_{35}$	-
2	?	$2.39F_{37}?$	+
3	$2.19G_{17}, 2.25G_{19}$	$2.20G_{37}$	-
4	$2.22H_{19}$	$2.3H_{39}$	+

In the sixth energy level (2.48 GeV) $n+m+k=5=2r_1+|m_1|+2r_2+|m_2|+2r_3+|m_3|$ which produces $|m_1|+|m_2|+|m_3|=5, 3, 1$ and $l_1+l_2+l_3 \geq 5, 3, 1$. We have then the possibilities

l_1, l_2, l_3	l	l_1, l_2, l_3	l	l_1, l_2, l_3	l	l_1, l_2, l_3	l
5,0,0	5	1,4,0	5,4,3	3,0,0	3	1,3,0	4,3,2
0,5,0	5	1,0,4	5,4,3	0,3,0	3	1,0,3	4,3,2
0,0,5	5	0,4,1	5,4,3	0,0,3	3	0,3,1	4,3,2
4,1,0	5,4,3	0,1,4	5,4,3	3,1,0	4,3,2	0,1,3	4,3,2
4,0,1	5,4,3			3,0,1	4,3,2		

Thus we identify the states

l	N	Δ	Parity
2	?	$2.39F_{37}$	+
3	?	$2.40G_{39}$	-
4	?	$2.42H_{3,11}$	+
5	$2.60I_{1,11}$?	-

The seventh energy state (2.79 GeV) has $n + m + k = 6 = 2r_1 + |m_1| + 2r_2 + |m_2| + 2r_3 + |m_3|$ which produces $|m_1| + |m_2| + |m_3| = 6, 4, 2, 0$ and $l_1 + l_2 + l_3 \geq 6, 4, 2, 0$. We have then the possibilities below

l_1, l_2, l_3	6,0,0	0,6,0	0,0,6	5,1,0	5,0,1	1,5,0	1,0,5
l	6	6	6	6,5,4	6,5,4	6,5,4	6,5,4

and the states

l	N	Δ	Parity
4	?	?	+
5	?	$2.75I_{3,13}$	-
6	$2.7K_{1,13}$	$2.95K_{3,15}$	+

3.2 Baryons Σ and Λ

Now let us do the calculation for Σ and Λ . According to Table 2 the first energy state (1.12 GeV) is ($n = 0, m = 0, k = 0$) and hence we can have $l_1 = 0, l_2 = 0, l_3 = 0$ which yields $l = 0$ and the states

l	Σ	Λ	Parity
0	$1.193P_{11}$	$1.116P_{01}$	+

In the second energy level (1.43 GeV) $n + m = 1, k = 0$ which makes $2r_1 + |m_1| + 2r_2 + |m_2| = 1$ and $2r_3 + |m_3| = 0$. This actually makes $|m_1| + |m_2| = 1$ and $|m_3| = 0$. That is, we have the condition $l_1 + l_2 \geq 1, l_3 \geq 0$ which allows us to choose the possibilities

l_1, l_2, l_3	1,1,0	1,0,1	0,1,1
l	2,1,0	2,1,0	2,1,0

that produce the states

l	Σ	Λ	Parity
0	$1.385P_{13}$?	+
1	?	$1.405S_{01}$	-
2	?	?	+

and the state 1.48Σ is either $S_{13}, S_{11}(l = 1)$ or $F_{15}(l = 2)$.

In the third energy level (1.62 GeV) $n = m = 0, k = 1$ and we have $|m_1| = 0, |m_2| = 0$ and $|m_3| = 1$. That is, we have the condition $l_1 \geq 0, l_2 \geq 0, l_3 \geq 1$ which allows us to choose $l_1 = l_2 = 0, l_3 = 1; l_1 = l_3 = 1, l_3 = 0; l_1 = 0, l_2 = l_3 = 1$, and the states

l	Σ	Λ	Parity
0	$1.66P_{11}$	$1.60P_{01}$	+
1	$1.62S_{11}$ $1.58D_{13}$	$1.67S_{01}$ $1.52D_{03}$	-
2	?	?	+

and then the state 1.56Σ is probably $F_{15}(l = 2)$.

The fourth energy level (1.74 GeV) has $n + m = 2 = 2r_1 + |m_1| + 2r_2 + |m_2|$ and $k = 2r_3 + |m_3| = 0$, and thus we obtain $|m_1| + |m_2| = 2, 0$ and $|m_3| = 0$. Hence we have the condition $l_1 + l_2 \geq 2, 0$ and $l_3 \geq 0$. We can then choose $l_1 = 2, l_2 = l_3 = 0; l_1 = l_3 = 0, l_2 = 2; l_1 = l_2 = 1, l_3 = 0$ and thus we can identify the states

l	Σ	Λ	Parity
0	$1.77P_{11}$	$1.81P_{01}$	+
1	$1.75S_{11}$ $1.67D_{13}, 1.775D_{15}$	$1.80S_{01}$ $1.69D_{03}$	-
2	?	$1.82F_{05}$	+

and then the level 1.69Σ is probably $F_{15}(l=2)$.

In the fifth energy level (1.93 GeV) $n+m=1=2r_1+|m_1|+2r_2+|m_2|$ and $k=1=2r_3+|m_3|$, and thus we obtain $|m_1|+|m_2|=1$ and $|m_3|=1$. Hence we have the condition $l_1+l_2\geq 1$ and $l_3\geq 1$. We can then have the sets $l_1=1, l_2=0, l_3=1$; $l_1=0, l_2=1, l_3=1$. Both yield $l=2, 1, 0$ and we can identify the states

l	Σ	Λ	Parity
0	$1.84P_{11}, 1.84P_{13}$	$1.89P_{03}$	+
1	$1.94D_{13}$	$1.83D_{05}$	-
2	$1.915F_{15}$?	+

The sixth energy level (2.05 GeV) has $n+m=3=2r_1+|m_1|+2r_2+|m_2|$ and $k=0=2r_3+|m_3|$, and thus we obtain $|m_1|+|m_2|=3, 1$ and $|m_3|=0$. Hence we have the condition $l_1+l_2\geq 3, 1$ and $l_3\geq 0$. We can then have the sets $l_1=2, l_2=1, l_3=0$; $l_1=1, l_2=2, l_3=0$ which make $l=3, 2, 1$, for $l_1+l_2\geq 3$ and the sets $l_1=1, l_2=1, l_3=0$; $l_1=1, l_2=1, l_3=0$ which make $l=2, 1, 0$, for $l_1+l_2\geq 1$. We can identify the states

l	Σ	Λ	Parity
0	$2.08P_{13}$?	+
1	$2.00S_{11}$?	-
2	$2.07F_{15}, 2.03F_{17}$	$2.02F_{07}$	+
3	?	?	-

In the seventh energy level (2.12 GeV) $n=0=2r_1+|m_1|, m=0=2r_2+|m_2|, k=2=2r_3+|m_3|$ and thus $|m_1|=0, |m_2|=0, |m_3|=2, 0$. Hence we have the condition $l_1\geq 0, l_2\geq 0$ and $l_3\geq 2, 0$. We can then choose the sets $l_1=0, l_2=0, l_3=2$; $l_1=0, l_2=0, l_3=3$ which make $l=3, 2$, and the states

l	Σ	Λ	Parity
$l=2$?	$2.11F_{05}$	+
$l=3$	$2.10G_{17}$	$2.10G_{07}$	-

Unfortunately, the angular momenta of the other energy levels have not been found but they can surely be explained according to what was developed above.

3.3 Baryons Ξ

For these baryons only some angular momenta are known. The first energy level (1.31 GeV) has $n = 0, m = 0, l = 0$ which make $l_1 = l_2 = l_3 = 0$ and $l = 0$ and is thus a P state. Therefore we obtain

l	Ξ	Parity
0	1.318 P_{11}	+

In the second energy level (1.62 GeV) $n = 1 = 2r_1 + |m_1|, m = 0 = 2r_2 + |m_2|, k = 0 = 2r_3 + |m_3|$ and thus $|m_1| = 1, |m_2| = 0, |m_3| = 0$. Hence we have the condition $l_1 \geq 1, l_2 \geq 0$ and $l_3 \geq 0$. We can then have the sets $l_1 = 1, l_2 = 0, l_3 = 0; l_1 = 1, l_2 = 1, l_3 = 0$ which make $l = 2, 1, 0$, and the states

l	Ξ	Parity
0	1.53 P_{13}	+
1	?	-
2	?	+

and thus the two levels 1.62 and 1.69 are probably either S, D or F states.

The third energy level (1.81 GeV) has $n = 0 = 2r_1 + |m_1|, m + k = 1 = 2r_2 + |m_2| + 2r_3 + |m_3|$ and thus $|m_1| = 0, |m_2| + |m_3| = 1$. Hence we have the condition $l_1 \geq 0, l_2 + l_3 \geq 1$. We can then have the sets $l_1 = 0, l_2 = 1, l_3 = 0; l_1 = 0, l_2 = 0, l_3 = 1$ which make $l = 1$, and the state

l	Ξ	Parity
$l = 1$	1.82 D_{13}	-

In the fourth energy level (1.93 GeV) $n = 2 = 2r_1 + |m_1|, m = 0 = 2r_2 + |m_2|, k = 0 = 2r_3 + |m_3|$ and thus $|m_1| = 2, |m_2| = 0, |m_3| = 0$. Hence we have the condition $l_1 \geq 0, 2, l_2 \geq 0$ and $l_3 \geq 0$. We can then choose the set $l_1 = 2, l_2 = 0, l_3 = 0$ which produces $l = 2$, and the state 1.93 GeV is probably an F state.

3.4 Relation between energy and angular momentum

From Eqs. 15 and 20 we have

$$E = (2r_1 + |m_1| + 1)h\nu_1 + (2r_2 + |m_2| + 1)h\nu_2 + (2r_3 + |m_3| + 1)h\nu_3$$

$$l_1 + l_2 + l_3 \geq l \geq ||l_1 - l_2| - l_3| \text{ with } l_1 \geq |m_1|, l_2 \geq |m_2|, l_3 \geq |m_3|.$$

in which l_1 , l_2 and l_3 are the quantum numbers of the angular momenta \vec{L}_1 , \vec{L}_2 , and \vec{L}_3 , and m_1, m_2, m_3 are the quantum numbers of their projections on the Z axis, respectively. Therefore, we clearly see that levels with large energies have large angular momenta as is quite evident from the experimental data.

4 The sizes of baryons

The solution in Cartesian coordinates is also useful for calculating in a quite simple manner the average size of a baryon. As is known the average potential energy of each oscillator is half of the total energy, that is,

$$\langle 12k\xi_i^2 \rangle = h\nu_i 2(v_i + 1/2) \quad (17)$$

but since there are two directions for each quark in the plane there actually are two oscillators per quark and thus we have the potential energy E_q associated to each quark

$$E_q = h\nu_q(n_i + 1/2) \quad (18)$$

where $n_i = 0, 1, 2, 3, \dots$ and $h\nu_q$ is the constituent quark mass constant. Thus taking into account Eq. 06 and the above fact on the relation between the total energy and the potential energy for an oscillator it can be written that

$$\begin{aligned} E(n, m, k) &= h\nu_1(n + 1/2) + h\nu_2(m + 1/2) + h\nu_3(k + 1/2) = \\ &= 2 \times (\langle 12k_1\eta_1^2 \rangle + \langle 12k_2\eta_2^2 \rangle + \langle 12k_3\eta_3^2 \rangle) = \\ &= \langle k_1\eta_1^2 \rangle + \langle k_2\eta_2^2 \rangle + \langle k_3\eta_3^2 \rangle \end{aligned} \quad (19)$$

where $\eta_i^2 = \xi_{ij}^2 + \xi_{ik}^2$ in which j and k are the two orthogonal directions of the two oscillators. One can then make the association

$$h\nu_i(n + 1/2) = \langle k_i\eta_i^2 \rangle \quad (20)$$

and hence the average radius \mathcal{R} of a baryon can given by

$$\begin{aligned} \mathcal{R}(n, m, k) &= \left(\sqrt{\langle \eta_1^2 \rangle \langle \eta_2^2 \rangle \langle \eta_3^2 \rangle} \right)^{1/3} = \\ &= \left(\sqrt{h\nu_1(n + 1/2)k_1 h\nu_2(m + 1/2)k_2 h\nu_3(k + 1/2)k_3} \right)^{1/3}. \end{aligned} \quad (21)$$

It is quite obvious that the application of the above formula should be first done to the proton. In the fundamental level $n = m = k = 0$ and $h\nu_1 = h\nu_2 = h\nu_3 = 0.31\text{GeV}$, and making the reasonable supposition that $k_1 = k_2 = k_3 = k$, thus

$$\mathcal{R}_0 = \mathcal{R}(0, 0, 0) = \sqrt{h\nu_1 k}. \quad (22)$$

If one uses for the average size of a proton the figure of $\sqrt{0.72}\text{fm} = 0.85\text{fm}$ [14] one has $k \approx 0.5\text{GeV}/\text{fm}^2$ which is a very reasonable figure because if it is multiplied by the characteristic distance of 1fm (of course) the constant $k' \approx 0.5\text{GeV}/\text{fm}$ is obtained which is quite close to the value of the constant K used in the QCD motivated potential [15], [16]

$$V_{QCD} = -CF\alpha_s r + Kr \quad (23)$$

which is assumed to be of the order of $1\text{GeV}/\text{fm}$.

From Table 1 one has that for $n = m = k = 2$ the energy of a proton is about 2.80GeV which gives an average radius of about 1.39fm and hence one sees that the size of a baryon does not change much with its energy. Therefore it can be said that the smallest radius of a proton is about 0.8fm and the largest radius is of the order of 1.4fm .

For the ground states of Σ^- and Ξ^- reference [15] gives, respectively, the radii $\sqrt{0.54}\text{fm} = 0.73\text{fm}$ and $\sqrt{0.43}\text{fm} = 0.66\text{fm}$. In terms of quarks Σ^- is *dus* and therefore one should have $k_2 = k_3$ and $k_1 \approx k \approx 0.5\text{GeV}/\text{fm}^2$ and

$$\mathcal{R}(n, m, k) = \left(\sqrt{h\nu_1(n+1) \times h\nu_2(m+1) \times h\nu_3(k+1) k_1 (k_3)^2} \right)^{1/3}. \quad (24)$$

Using the above value it is obtained that

$$0.73 = \left(\sqrt{0.31 \times 0.31 \times 0.505 \times (k_3)^2} \right)^{1/3}$$

which yields $k_3 = 0.80\text{GeV}/\text{fm}^2$. From Table 2 it is seen that for $n + m = 5$, $k = 1$, the energy is 3.17GeV which is the highest energy level up to now. If one takes, for example, $n = 2$, $m = 3$ one has an average radius of about 1.24fm . Also it is found that the average radii of Σ^- are much smaller than those of the proton for levels with the same quantum numbers n, m, k .

Now one can turn to Ξ^- which in terms of quarks is *dss*. Then it is expected to have the same $k_3 \approx 0.80\text{GeV}/\text{fm}^2$ (two of them) above and a new k , which

can be called k_{ss} . Using the ground state radius of Ξ^- (0.66fm) one obtains $k_{ss} \approx 1.47\text{GeV}/\text{fm}^2$. For the excited states the average radius (in fm) is thus

$$\mathcal{R}(n, m, k) = \left(\sqrt{0.31(n+1) \times 0.5(m+1) \times 0.5(k+1) 1.47(0.8)^2} \right)^{1/3} \quad (25)$$

which for the highest known excited state 2.55GeV ($n = 4, m = k = 0$) gives $\mathcal{R}(4, 0, 0) \approx 1.48\text{fm}$. Using the value $k_{ss} \approx 1.47\text{GeV}/\text{fm}^2$ the radius of the ground state of Ω is estimated to be about 0.58fm.

Putting together the above values the very important table below (Table 8) is obtained for the constant k (which is a sort of constant of confinement) in terms of the pairs of interacting quarks.

The table shows that k increases with the reduced mass of the pair of interacting quarks. When the data are fitted to a polynomial up to second order in the reduced mass of the pair of interacting quarks the following polynomial is obtained

$$k(\mu) = 0.1188 - 1.7561\mu + 28.6508\mu^2. \quad (26)$$

It is interesting that the coefficient of the last term is quite large and thus the first derivative increases very rapidly with μ . As more massive quarks are considered the degree of the polynomial may increase but just to have a lower bound one can calculate the value of k for the interaction between two top quarks. The above formula gives $k = 7416\text{GeV}/\text{fm}^2$. If the above data are fitted to a polynomial with a higher degree, for example, $k(\mu) = A + B\mu^2 + C\mu^3$, the following values are obtained: $A = -0.0268$, $B = 23.3794$, and $C = 0.2278$. Since the value of C is small and B is of the same order of 28.6508, the first polynomial (Eq. 26) is a good approximation. If it is used for obtaining the k between quarks u and c one has $k(uc) \approx 1.53\text{GeV}/\text{fm}^2$, $k(sc) \approx 3.7\text{GeV}/\text{fm}^2$, $k(cc) \approx 19\text{GeV}/\text{fm}^2$. And then one has that the radii of the ground states of the charmed baryons Λ_c^+ , Σ_c^{++} , Σ_c^+ and Σ_c^0 are about

$$R_c \approx \left(\sqrt{0.31 \times 0.31 \times 1.70.5(1.53)^2} \right)^{1/3} \approx 0.7\text{fm}$$

which is not so small due to the influence of the interaction between the two u quarks. As to Ω_c its ground state has a radius

$$R_{ssc} \approx \left(\sqrt{0.5 \times 0.5 \times 1.71.53(3.7)^2} \right)^{1/3} \approx 0.5\text{fm}$$

and the ground state of the ccc baryon has the quite small radius of just

$$R_{ccc} \approx \sqrt{1.719} \approx 0.3\text{fm}.$$

Since the value of $k(cc) \approx 19\text{GeV}/\text{fm}^2$ was obtained by means of an extrapolation the above figure of R_{ccc} should be taken as a crude approximation.

In the case of the $t\bar{t}t$ baryon an even cruder number is gotten for its radius because its value for k is expected to be larger than the above figure of $7416\text{GeV}/\text{fm}^2$, but it is instructive anyway to calculate its order of magnitude which in this case produces an upper bound for its radius. Therefore one can say that the radius of the ground state of the $t\bar{t}t$ system

$$R_{t\bar{t}t} < \sqrt{1747416} = 0.15\text{fm}.$$

which is a very important number just because the top quark is the most massive quark.

Since in this work the motion of the plane where quarks are sitting was not taken into account conclusions can not be drawn on the shape of baryons using the above figures.

5 On the spin-orbit interaction

We clearly notice that the splittings of some levels are caused by the spin-orbit interaction. For example, consider the states $1.90S_{31}$ and $1.91P_{31}$ of Δ which differ by the values of $l = 1$ and $l = 0$, respectively. Since we are assuming a harmonic potential and as the spin-orbit term is proportional to $1rdVdr$ we can approximately write

$$\Delta E_{SL} \approx C < \vec{S} \cdot \vec{L} > = C [j(j+1) - l(l+1) - s(s+1)] \quad (27)$$

for N and Δ baryons which have quarks with equal masses. Using for the above case $j = 1/2, s = 1/2$ we find $C \approx 5\text{MeV}$ which shows that the influence of the spin-orbit interaction is small. Considering the levels $1.91P_{31}$ and $1.92P_{33}$ we find $C \approx 3.3\text{MeV}$ which is of the same order of the above C . The same holds in the case of the other baryons: for example, consider the states $1.75S_{11}$ and $1.77P_{11}$ of Σ or the states $1.80S_{01}$ and $1.81P_{01}$ of Λ . We see that there is a small energy difference between these states.

6 Conclusion

The simple model presented above which considers that a baryon is composed of three nonrelativistic quarks produces very important results. First of all it describes quite well almost all energy levels of baryons with the proper assignment of the states as to parity and angular momentum. And the calculation also yields some reasonable figures for the sizes of baryons and for the important harmonic oscillator constant k which is directly related to confinement.

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